

Limit theorems for the painting of graphs by clusters*

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Abstract

We consider a generalization of the so-called **divide and color model** recently introduced by Häggström . We investigate the behaviour of the magnetization in large boxes and its fluctuations. Thus, laws of large numbers and Central Limit theorems are proved, both quenched and annealed. We show that the properties of the underlying percolation process roughly influence the behaviour of the coloring model. In the subcritical case, the limit magnetization is deterministic and the Central Limit Theorem admits a Gaussian limit. A contrario, the limit magnetization is not deterministic in the supercritical case and the limit of the Central Limit Theorem is not Gaussian, except in the particular model with exactly two colors which are equally probable.

AMS Classifications: 60K35, 82B20, 82B43.

KEY-WORDS: percolation, coloring model, law of large number, central limit theorem.

1 Introduction

The aim of this paper is to give some results concerning a pretty and natural model for the dependent coloring of vertices of a graph. This model has been recently introduced by Häggström [Häg00], who presented the first results, concerning essentially the presence (or absence) of percolation and the quasilocality properties. The model is easily described: choose a graph at random according to bond percolation, and then paint randomly and independently the different clusters, each cluster being monochromatic. There are several motivations for the study of such a model, the most relevant being its links with Ising or Potts models. We refer to the examples of the present article and to the introduction of Häggström's paper for detailed motivations.

*Preprint

In Häggström's model, the panel was constituted by a finite number of colours, which were chosen according to a measure ν on \mathbb{R} with finite support. For our purpose, the natural assumptions will only be the existence of a first or a second moment for ν .

Actually, we will study the mean magnetization in large boxes: we will identify the limit

$$M = \lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} X(x),$$

and determinate its variations: we will prove central limit theorems for quantities such that

$$\frac{1}{(|\Lambda_n|)^{1/2}} \left(\sum_{x \in \Lambda_n} X(x) - |\Lambda_n| M \right).$$

There are several natural questions: when is M deterministic ? What is the influence of the underlying bond percolation ? When is there convergence to a normal law in the Central Limit Theorem ?

These questions can be asked in two different approach. We shall use here the vocabulary usually used in the theory of random media.

- The quenched point of view: limit theorems are formulated once the graph has been (randomly) fixed.
- The annealed point of view: limit theorems are formulated under the randomization of the graph.

Indeed, we will show that the properties of the underlying percolation process roughly influence the behaviour of the coloring model. In the subcritical case, the limit magnetization is deterministic and the Central Limit Theorem admits a Gaussian limit. A contrario, the limit magnetization is not deterministic in the supercritical case and the limit of the Central Limit Theorem is not Gaussian, except in the particular model with exactly two colors which are equally probable. As examples, we will study the case where ν is "+/-" valued and the case where ν is a Gaussian measure.

2 Notations

We will deal here with stochastic processes indexed by \mathbb{Z}^d . Their definition will be related to some subgraphs of the d -dimensional cubic lattice \mathbb{L}^d , which is defined by $\mathbb{L}^d = (\mathbb{Z}^d, E_d)$, where $E_d = \{\{x, y\} \subset \mathbb{Z}^d; \sum_{i=1}^d |x_i - y_i| = 1\}$. In the following, the expression "subgraph of \mathbb{L}^d " will always be employed for each graph of the form $G = (\mathbb{Z}^d, E)$ where E is a subset of E_d . We say that two vertices $x, y \in \mathbb{Z}^d$ are adjacent in G if $\{x, y\} \in E$. Two vertices $x, y \in \mathbb{Z}^d$ are said to be connected in G if one can find a sequence of vertices containing x and y such that each element of the sequence is connected in G with the next one. A subset C of \mathbb{Z}^d is said to be connected if each pair of vertices in C are connected. The maximal connected sets are called the connected components. They partition \mathbb{Z}^d . The connected component of x and is denoted by $C(x)$. Note that the connected components are also called clusters. Conversely, a

subset D of \mathbb{Z}^d is said to be independent if no pair in D is constituted by adjacent vertices.

We will consider here subgraphs of \mathbb{L}^d which are generated by Bernoulli bond percolation on \mathbb{L}^d . Thus, we will denote by μ_p the image measure of $(\{0, 1\}^{E_d}, \mathcal{B}(\{0, 1\}^{E_d}), ((1-p)\delta_0 + p\delta_1)^{\otimes E_d})$ by

$$x \mapsto (\mathbb{Z}^d, \{e \in E_d; x_e = 1\}),$$

where $p \in (0, 1)$.

Let us choose a graph G at random under μ_p and recall the definition of some basic objects in percolation theory.

- The probability that 0 belongs to an infinite cluster:
 $\theta(p) = \mu_p(|C(0)| = +\infty)$.
- The critical probability:
 $p_c = \inf\{p \in (0, 1); \theta(p) > 0\}$.
- The mean size of a finite cluster:
 $\chi^f(p) = \sum_{k=1}^{+\infty} k \mu_p(|C(0)| = k)$.
- The number of open clusters per vertex
 $\kappa(p) = \sum_{k=1}^{+\infty} k^{-1} \mu_p(|C(0)| = k)$.

The following results will be currently used:

- μ_p is translation-invariant. As it is isomorphic to $(\{0, 1\}^{E_d}, \mathcal{B}(\{0, 1\}^{E_d}), ((1-p)\delta_0 + p\delta_1)^{\otimes E_d})$, its tail σ -field is trivial and the ergodic theorem can be employed with full power.
- If $p \in (0, p_c)$, then G contains no infinite cluster.
- If $p \in (p_c, 1)$, then G contains μ_p almost surely one unique infinite cluster.
- If $p \neq p_c$, then $\chi^f(p) < +\infty$.

If G is a subgraph of \mathbb{L}^d and if ν is a probability measure on \mathbb{R} , we will define the color-probability $P^{G, \nu}$ as follows: $P^{G, \nu}$ is the only measure on $(\mathbb{R}^{\mathbb{Z}^d}, \mathcal{B}(\mathbb{R}^{\mathbb{Z}^d}))$ under which the canonical projections X_i – defined, as usually by $X_i(\omega) = \omega_i$ – satisfy

- For each $i \in \mathbb{Z}^d$, the law of X_i is ν .
- For each independent set $S \subset \mathbb{Z}^d$, the variables $(X_i)_{i \in S}$ are independent.
- For each connected set $S \subset \mathbb{Z}^d$, the variables $(X_i)_{i \in S}$ are identical.

The randomized color-measure is defined by

$$P^{p, \nu} = \int P^{G, \nu} d\mu_p(G).$$

We also note $\Lambda_n = \{-n, \dots, n\}^d$.

3 Laws of large numbers

3.1 Quenched Law of large numbers

Theorem 1. *Let ν be a probability measure on \mathbb{R} with a first moment. We put $m = \int_{\mathbb{R}} x \, d\nu(x)$. Let $p \in (0, 1) \setminus \{p_c\}$*

For μ_p almost G , we have the following result :

$$\lim_{n \rightarrow +\infty} \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} X(x) = (1 - \theta(p))m + \theta(p)Z \quad P^{G, \nu} \text{ almost surely,}$$

where Z is the value taken by $X()$ along the infinite component if it exists, and 0 else.

Proof. The following lemma will be of higher importance in the following.

Lemma 1. *For each subgraph G of \mathbb{L}^d , let us denote by $(A_i)_{i \in I}$ the partition of G in connected component.*

Then, if $p \neq p_c$, we have for μ_p almost G , we have

$$\lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \sum_{i \in I; |A_i| < +\infty} |A_i \cap \Lambda_n|^2 = \chi^f(p),$$

where

$$\chi^f(p) = \sum_{k=1}^{+\infty} k P(C(0) = k).$$

Proof. Let us define $C'(x)$ by

$$C'(x) = \begin{cases} C(x) & \text{if } |C(x)| < +\infty \\ \emptyset & \text{else} \end{cases}$$

and $C'_n(x) = C'(x) \cap \Lambda_n$.

It is easy to see that

$$\sum_{i \in I; |A_i| < +\infty} |A_i \cap \Lambda_n|^2 = \sum_{x \in \Lambda_n} |C'_n(x)|. \quad (1)$$

We have $C'_n(x) \leq C(x)$, and the equality holds if and only if $C'(x) \subset \Lambda_n$.

The quantity residing in connected components intersecting the boundary of Λ_n can be controlled using well-known results about the distribution of the size of finite clusters. In both subcritical case and supercritical case, we can found $K > 0$ and $\beta > 0$ such that

$$P(+\infty > |C(x)| \geq n) \leq \exp(-Kn^\beta).$$

(We can take $\beta = 1$ when $p < p_c$ and $\beta = (d-1)/d$ if $p > p_c$. See for example the reference book of Grimmett [Gri99] for a detailed historical bibliography.) It follows

from a standard Borel-Cantelli argument that for μ_p almost G , there exists a (random) N such that

$$\forall n \geq N \quad \max_{x \in \Lambda_n} |C'(x)| \leq (\ln n)^{2/\beta}.$$

It follows that for each $x \in \Lambda_{n-(\ln n)^{2/\beta}}$, $C'(x)$ is completely inside Λ_n , and therefore $C'(x) = C'_n(x)$. Then,

$$\begin{aligned} \sum_{x \in \Lambda_{n-(\ln n)^{2/\beta}}} |C'(x)| &\leq \sum_{x \in \Lambda_n} |C'_n(x)| \leq \sum_{x \in \Lambda_n} |C'(x)| \\ \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_{n-(\ln n)^{2/\beta}}} |C'(x)| &\leq \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} |C'_n(x)| \leq \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} |C'(x)| \end{aligned}$$

By the ergodic theorem, we have μ_p almost surely

$$\lim_{n \rightarrow +\infty} \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} |C'(x)| = \mathbb{E} |C'(0)| = \chi^f(p).$$

Since $\lim_{n \rightarrow +\infty} \frac{|\Lambda_{n-(\ln n)^{2/\beta}}|}{|\Lambda_n|} = 1$, the result follows. \square

Remark If we forget technical controls, the key point of this proof is the identity (1). It is interesting to note that Grimmett [Gri99] used an analogous trick to prove that

$\lim_{n \rightarrow +\infty} k(n)/|\Lambda_n| = \kappa(p)$ almost surely, when $k(n)$ is the number of open clusters in Λ_n .

Let $(a_i)_{i \geq 1}$ be a sequence such that for each $x \in \mathbb{Z}^d$, there exists exactly one a_i connected to x in G . Then

$$\sum_{x \in \Lambda_n} X(x) = \sum_{i=1}^{+\infty} |C'_n(a_i)| X(a_i) + Z|\Lambda_n \cap I|, \quad (2)$$

where I is the infinite connected component (\emptyset if there is none). Since

$$|\Lambda_n \cap I| = \sum_{x \in \Lambda_n} \mathbb{1}_{|C(x)|=+\infty},$$

it follows from the ergodic theorem that

$$\lim_{n \rightarrow +\infty} \frac{1}{|\Lambda_n|} |\Lambda_n \cap I| = \mathbb{E} \mathbb{1}_{|C(0)|=+\infty} = \theta(p) \quad (3)$$

for μ_p almost G . From now on, we will suppose that G is a such a graph, and that, moreover, it is such that the conclusions of lemma 1 hold – μ_p almost all graph is such that.

Now, our goal is to apply a light improvement of the well-known proof of Etemadi [Ete81] for the law of large number. Let us state our result.

Proposition 1. *Let $(X_n)_{n \geq 1}$ be a sequence of pairwise independent and identically distributed variables. We suppose that X_1 is integrable and note $m = \mathbb{E} X_1$. Let also be $(\alpha_{i,n})_{i,n \in \mathbb{Z}_+}$ be a doubly indexed sequence of non-negative numbers such that*

•

$$\forall n \in \mathbb{Z}_+ \quad k(n) = |\{i \in \mathbb{Z}_+; \alpha_{i,n} \neq 0\}| < +\infty.$$

•

$\forall i \in \mathbb{Z}_+ \quad$ the sequence $(\alpha_{i,n})_{n \geq 1}$ is a non-decreasing converging sequence.

•

$$\frac{\sum_{i=1}^{+\infty} \alpha_{i,n}^2}{(\sum_{i=1}^{+\infty} \alpha_{i,n})^2} = O\left(\frac{1}{k(n)}\right).$$

•

$$\exists A > 0, d \geq 1 \quad k(n) \sim An^d.$$

•

$$\exists B > 0, e > 0 \quad \sum_{i=1}^{+\infty} \alpha_{i,n} \sim Bn^e.$$

Then, almost surely

$$\lim_{n \rightarrow +\infty} \frac{\sum_{i=1}^{+\infty} \alpha_{i,n} X_i}{\sum_{i=1}^{+\infty} \alpha_{i,n}} = m.$$

Proof. By linearity, it is sufficient to prove the theorem for nonnegative random variables. Assume then that $X_n \geq 0$. Moreover, we can assume without loss that $\alpha_{i,n} = 0$ for $i > k(n)$. This can be done by permuting columns of the matrix $(\alpha_{i,n})$. Let $d(n) = \sum_{i=1}^{+\infty} \alpha_{i,n}$. We define $S_n = \sum_{i=1}^{k(n)} \alpha_{i,n} X_i$ and $Q_n = S_n/d_n$. We also consider the truncated variables $Y_i = X_i \mathbb{1}_{X_i \leq k(i)}$ and associated sums and quotients: $S_n^* = \sum_{i=1}^{k(n)} \alpha_{i,n} Y_i$ and $Q_n^* = S_n^*/d_n$.

$$\begin{aligned} \text{Var } S_n^* &= \sum_{i=1}^{k(n)} \alpha_{i,n}^2 \text{Var } Y_i \\ &\leq \sum_{i=1}^{k(n)} \alpha_{i,n}^2 \mathbb{E} Y_i^2 \\ &\leq \sum_{i=1}^{k(n)} \alpha_{i,n}^2 \mathbb{E} X_i^2 \mathbb{1}_{X_i \leq k(i)} \\ &\leq \left(\sum_{i=1}^{k(n)} \alpha_{i,n}^2 \right) \mathbb{E} X_1^2 \mathbb{1}_{X_1 \leq k(n)} \end{aligned}$$

It follows that there exists $K > 0$ such that

$$\forall n \geq 1 \quad \text{Var } Q_n^* \leq K \frac{1}{k(n)} \mathbb{E} X_1^2 \mathbb{1}_{X_1 \leq k(n)}.$$

Now, fix $\beta > 1$ and define u_n to be the integer which is the closest to β^n . Then

$$\sum_{n=1}^{+\infty} \text{Var } Q_{u_n} \leq K \mathbb{E} X_1^2 \sum_{n=1}^{+\infty} \frac{1}{k(u_n)} \mathbb{1}_{X_1 \leq k(u_n)}$$

But since $k(u_n) \sim A\beta^{nd}$, it is easy to prove that

$$\sum_{n=N}^{+\infty} \frac{1}{k(u_n)} = O\left(\frac{1}{k(u_N)}\right).$$

Then, there exists $C > 0$ such that

$$\forall N \geq 1 \quad \sum_{n=N}^{+\infty} \frac{1}{k(u_n)} \leq C \frac{1}{k(u_N)}.$$

Now,

$$\begin{aligned} \sum_{n=1}^{+\infty} \text{Var } Q_{u_n}^* &\leq K \mathbb{E} X_1^2 \sum_{n: k(u_n) \geq X_1} \frac{1}{k(u_n)} \\ &\leq KC \mathbb{E} X_1^2 \frac{1}{k(\inf\{n; k(u_n) \geq X_1\})} \\ &\leq KC \mathbb{E} X_1^2 \frac{1}{X_1} = KC \mathbb{E} X_1 < +\infty \end{aligned}$$

It follows from Chebyshev's inequality and the first Borel-Cantelli lemma that

$$Q_{u_n}^* - \mathbb{E} Q_{u_n}^* \rightarrow 0 \text{ a.s.}$$

By monotone convergence,

$$\lim_{n \rightarrow +\infty} \mathbb{E} Y_n = \lim_{n \rightarrow +\infty} \mathbb{E} X_1 \mathbb{1}_{X_1 \geq k(n)} = \mathbb{E} X_1.$$

Let N be such that $|\mathbb{E} Y_k - \mathbb{E} X_1| \leq \varepsilon$ for $k > N$. Then, for $n \geq N$

$$|\mathbb{E} Q_{u_n} - \mathbb{E} X_1| \leq \frac{\mathbb{E} X_1}{d_n} \left(\sum_{i=1}^N \lim_{k \rightarrow +\infty} \alpha_{i,k} \right) + \varepsilon.$$

Then, $\overline{\lim}_{n \rightarrow +\infty} |\mathbb{E} Q_n^* - \mathbb{E} X_1| \leq \varepsilon$, and since ε is arbitrary, $\lim \mathbb{E} Q_n^* = \mathbb{E} X_1$. It follows that

$$Q_{u_n}^* \rightarrow \mathbb{E} X_1 \text{ a.s.}$$

Now, we go back to not truncated variables. Since $A' = \sup_n n/k(n) < +\infty$, we have

$$\sum_{n=1}^{+\infty} P(X_n \neq Y_n) = \sum_{n=1}^{+\infty} P(X_n > k(n)) = \sum_{n=1}^{+\infty} P(X_1 > k(n)) \leq \sum_{n=1}^{+\infty} P(X_1 > A'n) \leq A' \mathbb{E} X_1 < +\infty$$

It follows that for almost all ω , there exists $n(\omega)$ such that $X_k(\omega) = Y_k(\omega)$ for $k \geq n(\omega)$. Then, for $n \geq n(\omega)$

$$|Q_n(\omega) - Q_n^*(\omega)| \leq \frac{1}{d_n} \sum_{i=1}^{n(\omega)} \left(\lim_{k \rightarrow +\infty} \alpha_{i,k} \right) X_i(\omega).$$

It follows that

$$Q_{u_n} \rightarrow \mathbb{E} X_1 \text{ a.s.}$$

If $u_n \leq k \leq u_{n+1}$, then since $(S_n)_{n \geq 1}$ is non-decreasing, we have

$$\frac{d_{u_n}}{d_{u_{n+1}}} Q_{u_n} \leq Q_k \leq \frac{d_{u_{n+1}}}{d_{u_n}} Q_{u_{n+1}}.$$

Since $\frac{d_{u_{n+1}}}{d_{u_n}} \rightarrow \beta^e$, it follows that

$$\frac{1}{\beta^e} \mathbb{E} X_1 \leq \underline{\lim}_{k \rightarrow +\infty} Q_k \leq \overline{\lim}_{k \rightarrow +\infty} Q_k \leq \beta^e \mathbb{E} X_1$$

Since this is true for each $\beta > 1$, we have proved that

$$\lim_{k \rightarrow +\infty} Q_k = \mathbb{E} X_1 \text{ a.s.}$$

□

Our goal is to apply this result to the sequence $(\alpha_{i,n})$ defined by $\alpha_{i,n} = |C'_n(a_i)|$. Since the sequences $(C'_n(x))_{n \geq 1}$ are non-decreasing, so are the sequences $(\alpha_{i,n})_{n \geq 1}$.

Indeed, we have $\lim_{n \rightarrow +\infty} \alpha_{i,n} = |C(a_i)| < +\infty$. Moreover,

$$\sum_{i=1}^{+\infty} \alpha_{i,n} = |\Lambda_n \setminus I|. \quad (4)$$

Since the $\alpha_{i,n}$'s are natural numbers, it follows that $k(n)$ is finite. In fact, $k(n)$ is the number of finite components which intersect Λ_n . Together with the ergodic theorem, (4) gives

$$\sum_{i=1}^{+\infty} \alpha_{i,n} \sim (1 - \theta(p)) |\Lambda_n| \sim (1 - \theta(p)) 2^d n^d. \quad (5)$$

As already mentioned, Grimmett has proved that

$$\lim_{n \rightarrow +\infty} k(n)/|\Lambda_n| = \kappa(p). \quad (6)$$

Then, we have

$$k(n) \sim \kappa(p) 2^d n^d \quad (7)$$

We must now prove that

$$\frac{\sum_{i=1}^{+\infty} \alpha_{i,n}^2}{\left(\sum_{i=1}^{+\infty} \alpha_{i,n}\right)^2} k(n)$$

is bounded. But

$$\frac{\sum_{i=1}^{+\infty} \alpha_{i,n}^2}{\left(\sum_{i=1}^{+\infty} \alpha_{i,n}\right)^2} k(n) \sim (1 - \theta(p))^{-2} \frac{\sum_{i=1}^{+\infty} \alpha_{i,n}^2}{|\Lambda_n|} \frac{k(n)}{|\Lambda_n|}$$

Using the conclusions of lemma 1 and (6), we get

$$\lim_{n \rightarrow +\infty} \frac{\sum_{i=1}^{+\infty} \alpha_{i,n}^2}{\left(\sum_{i=1}^{+\infty} \alpha_{i,n}\right)^2} k(n) = \frac{\chi^f(p)}{(1 - \theta(p))^2} \kappa(p),$$

which completes the checking of the assumptions. It follows that

$$\lim_{n \rightarrow +\infty} \frac{1}{|\Lambda_n \setminus I|} \sum_{i=1}^{+\infty} |C'_n(a_i)| X(a_i) = m \text{ a.s.} \quad (8)$$

Since $|\Lambda_n \setminus I| \sim (1 - \theta(p)) |\Lambda_n|$, it comes from (2),(3) and (8) that

$$\lim_{n \rightarrow +\infty} \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} X(x) = (1 - \theta(p))m + \theta(p)Z \quad P^{G,\nu} \text{ almost surely.}$$

□

We will now formulate an easy, but important corollary.

Corollary 1. • *Z is measurable with respect to the tail σ -field*

$$\mathcal{T} = \bigcap_{n \geq 1} \mathcal{F}_{\mathbb{Z}^d \setminus \Lambda_n},$$

where \mathcal{F}_S is the σ -field generated by the $(X_i)_{i \in S}$.

- *For $p > p_c$, \mathcal{T} is not trivial under $P^{G,\nu}$ for μ_p almost every G as soon as ν is not a Dirac measure.*

Proof. The first point is a consequence of the formula given in theorem 1 and the second point is a consequence of the first one, because Z is non constant as soon as ν is not a Dirac measure. □

The fact that Z is \mathcal{T} -measurable will be important for the formulation of annealed results, because the environment is forgotten once we have randomized under μ_p . Indeed, whereas the infinite component can not always be recovered, the value of $X()$ along this component can.

3.2 Annealed Law of large numbers

In this case, the annealed theorem is an easy consequence of the quenched one.

Corollary 2. *Let ν be a probability measure on \mathbb{R} with a first moment. We put $m = \int_{\mathbb{R}} x \, d\nu(x)$.*

Then, for $p \in (0, 1) \setminus \{p_c\}$:

$$\lim_{n \rightarrow +\infty} \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} X(x) = (1 - \theta(p))m + \theta(p)Z \quad P^{p,\nu} \text{ almost surely,}$$

where Z is the value taken by $X()$ along the infinite component if it exists, and 0 else.

Proof. Let $C = \{ \lim_{n \rightarrow +\infty} \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} X(x) = (1 - \theta(p))m + \theta(p)Z \}$. We have

$$P^{p,\nu}(C) = \int P^{G,\nu}(C) \, d\mu_p(G) = \int 1 \, d\mu_p(G) = 1,$$

since $P^{G,\nu}(C) = 1$ for μ_p almost G . \square

Of course, the existence of the annealed limit is not an hard result: since $P^{p,\nu}$ is invariant under the translations, the ergodic theorem ensures that

$$\lim_{n \rightarrow +\infty} \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} X(x) = \mathbb{E} [X_0 | \mathcal{T}] \text{ almost surely.} \quad (9)$$

Indeed, the annealed law of large number could be rephrased in

$$\mathbb{E} [X_0 | \mathcal{T}] = (1 - \theta(p))m + \theta(p)Z \quad (10)$$

Here is the annealed result analogous to corollary 1. It can be seen as a consequence of (10).

Corollary 3. *For $p > p_c$, \mathcal{T} is not trivial under $P^{p,\nu}$ as soon as ν is not a Dirac measure.*

3.3 Examples

3.3.1 ”+/-” valued spin system

It is the simplest models that we can study: only two values are taken: ”+1” and ”-1”, with probability α and $1 - \alpha$. In the terminology of Häggström [Häg99], it is denoted as the $r + s$ -state fractional fuzzy Potts model at inverse temperature $-\frac{1}{2} \ln(1 - p)$, with $r = \alpha$ and $s = 1 - \alpha$. This name refers to the fact that the fuzzy Potts model can be realized using random clusters by an analogous painting procedure. For more details, see Häggström [Häg99].

We can also remark that for μ_p almost G , we have

$$P^{G, (1-\alpha)\delta_{-1} + \alpha\delta_{+1}} = \alpha \lim_{\beta \rightarrow +\infty} \mathcal{I}_{G, \beta, h}^+ + (1-\alpha) \lim_{\beta \rightarrow +\infty} \mathcal{I}_{G, \beta, h}^-,$$

where $\lim_{\beta \rightarrow +\infty} \mathcal{I}_{G, \beta, h}^+$ (resp. $\lim_{\beta \rightarrow +\infty} \mathcal{I}_{G, \beta, h}^-$) is the Ising Gibbs measure on G at inverse temperature β with the external field $h = \frac{1}{2} \ln(\alpha/(1-\alpha))$ which is maximal (resp. minimal) for the stochastic domination. Thus,

$$P^{p, (1-\alpha)\delta_{-1} + \alpha\delta_{+1}} = \lim_{\beta \rightarrow +\infty} \int \alpha \mathcal{I}_{G, \beta, h}^+ + (1-\alpha) \mathcal{I}_{G, \beta, h}^- d\mu_p.$$

In this sense, we can say that $P^{p, (1-\alpha)\delta_{-1} + \alpha\delta_{+1}}$ arises at the zero temperature limit of an Ising model on a randomly diluted lattice. For precise definitions and results relative to Ising ferromagnets on random subgraphs generated by bond percolation, see Georgii [Geo81] and also the recent article of Häggström, Schonmann and Steif [HSS00].

If we choose $\nu = (1-\alpha)\delta_{-1} + \alpha\delta_{+1}$ with $\alpha \in (0, 1)$, it follows that the magnetization is

$$M = \lim_{n \rightarrow +\infty} \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} X(x) = \begin{cases} 2\alpha(1 - \theta(p)) + 2\theta(p) - 1 & \text{with probability } \alpha \\ 2\alpha(1 - \theta(p)) - 1 & \text{with probability } 1 - \alpha \end{cases} \quad (11)$$

When $p \in (0, p_c)$, the magnetization is deterministic. Moreover, it follows from (11) that the signum of the magnetization is deterministic if and only if

$$\max(\alpha, 1-\alpha)(1 - \theta(p)) \geq \frac{1}{2}.$$

Note that if $\theta(p) \geq \frac{1}{2}$ the signum of the magnetization can not be deterministic.

Note that in the case $\alpha = \frac{1}{2}$, the annealed law of the magnetization has been identified by Häggström ([Häg00] Proposition 2.1) using a spin-flip argument.

3.3.2 A quenched Gaussian system

Here we choose $\nu = \mathcal{N}(0, 1)$. For each G , $P^{G, \nu}$ is a Gaussian measure. Here, we have

$$M = \lim_{n \rightarrow +\infty} \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} X(x) = \theta(p)Z. \quad (12)$$

In other words, M is almost surely null when $p < p_c$ and $M \sim \mathcal{N}(0, \theta(p)^2)$ when $p > p_c$.

We emphasize that these large numbers theorems are valid both quenched and annealed. This will not more be so simple for Central Limit theorems.

4 Central Limit Theorems

4.1 Quenched Central Limit Theorem

Theorem 2. *Let ν be a probability measure on \mathbb{R} with a second moment. We put $m = \int_{\mathbb{R}} x \, d\nu(x)$ and $\sigma^2 = \int_{\mathbb{R}} (x - m)^2 \, d\nu(x)$.*

For μ_p almost G , we have the following results:

- The subcritical case
If $p \in (0, p_c)$, then

$$\lim_{n \rightarrow +\infty} \frac{1}{|\Lambda_n|^{1/2}} \left(\sum_{x \in \Lambda_n} (X(x) - m) \right) = \mathcal{N}(0, \chi^f(p) \sigma^2).$$

- The supercritical case
If $p \in (p_c, 1)$, then

$$\lim_{n \rightarrow +\infty} \frac{1}{|\Lambda_n|^{1/2}} \left(\sum_{x \in \Lambda_n \setminus I} (X(x) - m) \right) = \mathcal{N}(0, \chi^f(p) \sigma^2)$$

where I is the infinite component of G .

For simplicity, we will give the proof in the supercritical case – which contains the proof of the subcritical case.

Proof.

$$\sum_{x \in \Lambda_n \setminus I} (X(x) - m) = \sum_{i=1}^{+\infty} |C'_n(a_i)| (X(a_i) - m)$$

Then

$$\frac{1}{|\Lambda_n|^{1/2}} \sum_{x \in \Lambda_n \setminus I} (X(x) - m) = \left(\frac{s_n^2}{|\Lambda_n|} \right)^{1/2} \frac{1}{s_n} \sum_{i=1}^{+\infty} |C'_n(a_i)| (X(a_i) - m),$$

with

$$s_n^2 = \sum_{i=1}^{+\infty} |C'_n(a_i)|^2.$$

By lemma 1, we have for μ^p almost G $\lim_{n \rightarrow +\infty} \frac{s_n^2}{|\Lambda_n|} = \chi^f(p)$.

Now, we have just to prove

$$\frac{1}{s_n} \sum_{i=1}^{+\infty} |C'_n(a_i)| (X(a_i) - m) \Longrightarrow \mathcal{N}(0, \sigma^2). \quad (13)$$

Therefore, we will prove that for μ^p almost G , the sequence $Y_{n,k} = |C'_n(a_i)|(X(a_i) - m)$ satisfies the Lindeberg condition. For each $\varepsilon > 0$, we have

$$\begin{aligned} \sum_{k=1}^{+\infty} \frac{1}{s_n^2} \int_{|Y_{n,k}| \geq \varepsilon s_n} Y_{n,k}^2 dP^{G,\nu} &= \sum_{k=1}^{+\infty} \frac{|C'_n(a_k)|^2}{s_n^2} \int_{|C'_n(a_k)||x| \geq \varepsilon s_n} (x - m)^2 d\nu(x) \\ &\leq \int_{|x| \geq \frac{\varepsilon}{\eta_n}} (x - m)^2 d\nu(x), \end{aligned}$$

with $\eta_n = \frac{\sup_{k \geq 1} |C'_n(a_k)|}{s_n}$. Then, the Lindeberg condition is fulfilled if $\lim \eta_n = 0$. But we have already seen that $s_n \sim (\chi^f(p)|\Lambda_n|)^{1/2}$, whereas $\sup_{k \geq 1} |C'_n(a_k)| = O((\ln n)^{2/\beta})$. This concludes the proof. \square

4.2 Annealed Central Limit Theorem

Theorem 3. *Let ν be a probability measure on \mathbb{R} with a second moment. We put $m = \int_{\mathbb{R}} x d\nu(x)$ and $\sigma^2 = \int_{\mathbb{R}} (x - m)^2 d\nu(x)$. Let $p \in (p_c, 1)$. We emphasize that G is randomized under μ_p .*

- The subcritical case
If $p \in (0, p_c)$, then

$$\lim_{n \rightarrow +\infty} \frac{1}{|\Lambda_n|^{1/2}} \left(\sum_{x \in \Lambda_n} X(x) - m|\Lambda_n| \right) = \gamma$$

where

$$\gamma = \mathcal{N}(0, \chi^f(p)\sigma^2)$$

- The supercritical case
If $p \in (p_c, 1)$, then

$$\lim_{n \rightarrow +\infty} \frac{1}{|\Lambda_n|^{1/2}} \left(\sum_{x \in \Lambda_n} X(x) - ((1 - \theta(p))m + \theta(p)Z)|\Lambda_n| \right) = \gamma$$

where

γ is the image of $\mathcal{N}(0, \chi^f(p)\sigma^2) \times \mathcal{N}(0, \sigma_p^2) \times \nu$ by $(x, y, z) \mapsto x + y(z - m)$,

with

$$\sigma_p^2 = \sum_{k \in \mathbb{Z}^d} (P(0 \in I \text{ and } k \in I) - \theta(p)^2),$$

where I is the infinite component of G .

In the subcritical case, the Annealed Central Limit Theorem is a simple consequence of the Quenched Central Limit Theorem.

In order to prove this result in the supercritical case, we will need a Central Limit Theorem related to the variations of the size of the intersection of the infinite cluster with large boxes.

Proposition 2. *Under μ_p , we have*

$$\frac{|\Lambda_n \cap I| - \theta(p)|\Lambda_n|}{|\Lambda_n|^{1/2}} \Longrightarrow \mathcal{N}(0, \sigma_p^2),$$

where I is the infinite component of G .

Proof.

$$|\Lambda_n \cap I(\omega)| - \theta(p)|\Lambda_n| = \sum_{k \in \Lambda_n} f(T^k \omega),$$

where T^k is the translation operator defined by $T^k(\omega) = (\omega_{n+k})_{n \in \mathbb{Z}^d}$ and $f = \mathbb{1}_{\{|C(0)|=+\infty\}} - \theta(p)$. Moreover f is an increasing function and μ^p satisfies the FKG inequalities. Then, $(f(T^k \omega))_{k \in \mathbb{Z}^d}$ is a stationary random field of square integrable satisfying to the FKG inequalities. Therefore, according to Newman [New80], the Central Limit Theorem is true if we prove that the quantity

$$\sum_{k \in \mathbb{Z}^d} \text{Cov}(f, f \circ T^k)$$

is finite. But $\text{Cov}(f, f \circ T^k) = \text{Cov}(Y_0, Y_k)$, with $Y_k = \mathbb{1}_{|C(k)| < +\infty}$. Now,

$$Y_k = \sum_{n=0}^{+\infty} F_{n,k}, \text{ with } F_{n,k} = \mathbb{1}_{\{|C(k)|=n\}}.$$

Hence

$$\begin{aligned} \text{Cov}(Y_0, Y_k) &= \sum_{n=0}^{+\infty} \sum_{p=0}^{+\infty} \text{Cov}(F_{n,0}, F_{p,k}) \\ &= \sum_{n=0}^{+\infty} (\text{Cov}(F_{n,0}, F_{n,k}) + 2 \sum_{p=0}^{n-1} \text{Cov}(F_{n,0}, F_{p,k})) \\ &= \sum_{n > \|k\|/2-1}^{+\infty} (\text{Cov}(F_{n,0}, F_{n,k}) + 2 \sum_{p=0}^{n-1} \text{Cov}(F_{n,0}, F_{p,k})) \\ &= \sum_{n > \|k\|/2-1}^{+\infty} \text{Cov}(F_{n,0}, F_{n,k} + 2 \sum_{p=0}^{n-1} F_{p,k}), \end{aligned}$$

because $F_{n,0}$ and $F_{p,k}$ are independent as soon as $\|k\| \geq p + n + 2$. Since $F_{n,0} \geq 0$ and $0 \leq F_{n,k} + 2 \sum_{p=0}^{n-1} F_{p,k} \leq 2$, we have

$$|\text{Cov}(F_{n,0}, F_{n,k} + 2 \sum_{p=0}^{n-1} F_{p,k})| \leq 2\mathbb{E} F_{n,0} = 2P(|C(0)| = n).$$

Then, $|\text{Cov}(Y_0, Y_k)| \leq \sum_{n > \|k\|/2-1}^{+\infty} 2P(|C(0)| = n)$ and

$$\sum_{k \in \mathbb{Z}^d} \text{Cov}(f, f \circ T^k) \leq 2 \sum_{n=1}^{+\infty} |\Lambda_{2(n+1)}| P(|C(0)| = n).$$

Since Kesten and Zhang [KZ90] have proved the existence of $\eta(p) > 0$ such that

$$\forall n \in \mathbb{Z}_+ \quad P(|C(0)| = n) \leq \exp(-\eta(p)n^{(d-1)/d}),$$

it follows that the series converges. Of course, a so sharp estimate is not necessary for our purpose. Estimates derived from Chayes, Chayes and Newman [CCN87], and from Chayes, Chayes, Grimmett, Kesten and Schonmann [CCG⁺89] would have been sufficient. \square

Proof. Rearranging the terms of the sum, we easily obtain

$$\left(\sum_{x \in \Lambda_n} X(x) - ((1 - \theta(p))m + \theta(p)Z) |\Lambda_n| \right) = \sum_{x \in \Lambda_n \setminus I} (X(x) - m) + (Z - m)(|I \cap \Lambda_n| - |\Lambda_n| \alpha)$$

We will now put

$$Q_n = \frac{1}{|\Lambda_n|^{1/2}} \left(\sum_{x \in \Lambda_n} X(x) - ((1 - \theta(p))m + \theta(p)Z) |\Lambda_n| \right),$$

and define

$$\forall t \in \mathbb{R} \quad \phi_n(t) = \mathbb{E} \exp(iQ_n)$$

and

$$\forall t \in \mathbb{R} \quad \phi_{n,z}(t) = \mathbb{E} \exp(iQ_n) | \{Z = z\}$$

As usually, it means that $\mathbb{E} (\exp(iQ_t) | Z) = \phi_{n,z}(Z)$. It is also important to emphasize that the following properties are fulfilled under $P^{p,\nu}$:

- G is independent from Z .
- $(X_k \mathbb{1}_{k \notin I})_{k \in \mathbb{Z}^d}$ is independent from Z .

Therefore, we have

$$\phi_{n,z}(t) = \mathbb{E} \exp\left(-\frac{it}{|\Lambda_n|^{1/2}} \sum_{x \in \Lambda_n \setminus I} (X(x) - m) + (z - m)(|I \cap \Lambda_n| - |\Lambda_n| \alpha)\right)$$

Conditioning by $\sigma(G)$ and using the fact that I is $\sigma(G)$ -measurable, we get $\phi_{n,z}(t) = \mathbb{E} f_n(t, \cdot) g_n((z - m)t, \cdot)$, with

$$\begin{aligned} f_n(t, \omega) &= \mathbb{E} \exp\left(-\frac{it}{|\Lambda_n|^{1/2}} \sum_{x \in \Lambda_n \setminus I} (X(x) - m) \middle| \sigma(G)\right) \\ &= \int \exp\left(-\frac{it}{|\Lambda_n|^{1/2}} \sum_{x \in \Lambda_n \setminus I(\omega)} (X(x) - m) \right) dP^{G(\omega), \nu} \end{aligned}$$

and

$$g_n(t, \omega) = \exp\left(-\frac{it}{|\Lambda_n|^{1/2}} (|I(\omega) \cap \Lambda_n| - |\Lambda_n| \alpha)\right)$$

By theorem 2 we have for each $t \in \mathbb{R}$ and $P^{p, \nu}$ almost ω : $\lim_{n \rightarrow +\infty} f_n(t, \omega) = \exp(-\frac{t^2}{2} \chi^f(p) \sigma^2)$

Then, by dominated convergence

$$\lim_{n \rightarrow +\infty} \mathbb{E} (f_n(t, \cdot) - \exp(-\frac{t^2}{2} \chi^f(p) \sigma^2)) g_n((z - m)t, \cdot) = 0.$$

Then

$$\begin{aligned} \lim_{n \rightarrow +\infty} \mathbb{E} f_n(t, \cdot) g_n((z - m)t, \cdot) &= \lim_{n \rightarrow +\infty} \exp(-\frac{t^2}{2} \chi^f(p) \sigma^2) \mathbb{E} g_n((z - m)t, \cdot) \\ &= \exp(-\frac{t^2}{2} \chi^f(p) \sigma^2) \exp(-\frac{t^2}{2} (z - m)^2 \sigma_p^2) \end{aligned}$$

where the last equality follows from Proposition 2. We have just proved that

$$\lim_{n \rightarrow \infty} \phi_{n,z}(t) = \exp(-\frac{t^2}{2} (\chi^f(p) \sigma^2 + (z - m)^2 \sigma_p^2)).$$

Since $\phi_n(t) = \int \phi_{n,z}(t) d\nu(z)$, we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \phi_n(t) &= \int \exp(-\frac{t^2}{2} (\chi^f(p) \sigma^2 + (z - m)^2 \sigma_p^2)) d\nu(z) \\ &= \int \exp(itx) d\gamma(x). \end{aligned}$$

By the theorem of Levy , it follows that $Q_n \Rightarrow \gamma$. □

4.3 Examples

4.3.1 ”+/-” valued spin system

If we choose $\nu = (1 - \alpha)\delta_{-1} + \alpha\delta_1$ with $\alpha \in (0, 1)$, it follows that

- In the subcritical case $p \in (0, p_c)$, then

$$\gamma = \alpha \mathcal{N}(0, 4\alpha(1-\alpha)\chi^f(p)).$$

- In the supercritical case $p \in (p_c, 1)$, then

$$\gamma = \alpha \mathcal{N}(0, 4\alpha(1-\alpha)\chi^f(p) + 4(1-\alpha)^2\sigma_p^2) + (1-\alpha)\mathcal{N}(0, 4\alpha(1-\alpha)\chi^f(p) + 4\alpha^2\sigma_p^2).$$

Remarks

1. For the "+/-" valued spin system in the subcritical case, the annealed Central Limit Theorem can be simply proved without using the quenched one: since $\int \omega_k dP^{G,\nu} = m$ for each k and each G , it follows that the covariance of X_0 and X_k under $P^{p,\nu}$ is

$$\begin{aligned} \text{Cov}(X_0, X_k) &= \int \left(\int (\omega_0 - m)(\omega_k - m) dP^{G,\nu} \right) d\mu_p(G) \\ &= \int \sigma^2 \mathbb{1}_{\{k \in C(0)\}} d\mu_p(G) \\ &= \sigma^2 P(k \in C(0)) \end{aligned}$$

Then,

$$\begin{aligned} \sum_{k \in \mathbb{Z}^d} \text{Cov}(X_0, X_k) &= \sum_{k \in \mathbb{Z}^d} \int \sigma^2 \mathbb{1}_{\{k \in C(0)\}} d\mu_p(G) \\ &= \sigma^2 \int \sum_{k \in \mathbb{Z}^d} \mathbb{1}_{\{k \in C(0)\}} d\mu_p(G) \\ &= \sigma^2 \int |C(0)| d\mu_p(G) \\ &= \sigma^2 \chi(p), \end{aligned}$$

with $\chi(p) = \mathbb{E} |C(0)| = \chi^f(p) + \theta(p)(+\infty)$. $\sum_{k \in \mathbb{Z}^d} \text{Cov}(X_0, X_k)$ is a convergent series when $p < p_c$ and a divergent one else.

In the subcritical case, the theorem of Newman [New80] ensures that the Central Limit is valid as soon as the translation-invariant measure $P^{p,\nu}$ satisfy to the F.K.G. inequalities. Since Häggström and Schramm [Häg00] have proved the F.K.G. inequalities for the "+/-" valued spin system, we get a simple proof for the annealed Central Limit Theorem in this particular case.

2. In the case were $\alpha = \frac{1}{2}$, γ is a Gaussian measure as well in the subcritical case ($\gamma = \mathcal{N}(0, \chi^f(p))$) as in the supercritical case ($\gamma = \mathcal{N}(0, \chi^f(p) + \sigma_p^2)$). It provides an example where there is a classical Central Limit Theorem whereas

the "susceptibility" $\sum_{k \in \mathbb{Z}^d} \text{Cov}(X_0, X_k)$ is infinite.

It is the "only" case with a Gaussian limit in the supercritical case, as tell the following remark.

3. If $p \in (p_c, 1)$, then γ is Gaussian if and only if there exist $a, b \in \mathbb{R}$ such that $\nu = \frac{1}{2}(\delta_a + \delta_b)$.

Proof. Using the characteristic function, it is easy to see that γ is Gaussian if and only if $\gamma' = \int \mathcal{N}(0, (z-m)^2 \sigma_p^2)(t) d\nu(z)$ does. Let us define, for $k \in \mathbb{Z}_+$: $m_k = \int z^{2k} d\gamma(z)$. We have

$$\begin{aligned} m_k &= \int \mathcal{N}(0, (z-m)^2 \sigma_p^2)(x \mapsto x^{2k}) d\nu(z) \\ &= \int \frac{(2k)!}{k! 2^k} (z-m)^{2k} d\nu(z) \\ &= \frac{(2k)!}{k! 2^k} \int (z-m)^{2k} d\nu(z) \end{aligned}$$

By definition of γ' , γ' is a symmetric measure. So if γ' is Gaussian, it is centered and we have

$$\forall k \in \mathbb{Z}_+ \quad m_k = \frac{(2k)!}{k! 2^k} m_1^k.$$

Then, we have

$$\forall k \in \mathbb{Z}_+ \quad \int (z-m)^{2k} d\nu(z) = m_1^k.$$

If we denote by ν' the image of ν by $z \mapsto (z-m)^2$, we have

$$\forall k \in \mathbb{Z}_+ \quad \int_{\mathbb{R}_+} z^k d\nu'(z) = m_1^k$$

Then, we have

$$\text{supp ess } \nu' = \lim_{k \rightarrow +\infty} \left(\int_{\mathbb{R}_+} z^k d\nu'(z) \right)^{1/k} = m_1 = \int_{\mathbb{R}_+} z d\nu'(z)$$

It follow that for ν' almost z , $z = \text{supp ess } \nu'$: ν' is a Dirac measure. Therefore, $\text{supp } \nu \subset \{m - \sqrt{m_1}, m + \sqrt{m_1}\}$. Since $m = \int z d\nu(z)$, we necessary have $\nu(m - \sqrt{m_1}) = \nu(m + \sqrt{m_1}) = \frac{1}{2}$ and then $\nu = \frac{1}{2}(\delta_{m-\sqrt{m_1}} + \delta_{m+\sqrt{m_1}})$. \square

4.3.2 The quenched Gaussian system

In the case $\nu = \mathcal{N}(0, 1)$, theorem 3 takes the following form:

- *The subcritical case*

If $p \in (0, p_c)$, then

$$\lim_{n \rightarrow +\infty} \frac{1}{|\Lambda_n|^{1/2}} \left(\sum_{x \in \Lambda_n} X(x) \right) = \gamma$$

where

$$\gamma = \mathcal{N}(0, \chi^f(p) \sigma^2)$$

- *The supercritical case*

If $p \in (p_c, 1)$, then

$$\lim_{n \rightarrow +\infty} \frac{1}{|\Lambda_n|^{1/2}} \left(\sum_{x \in \Lambda_n} X(x) - \theta(p) Z |\Lambda_n| \right) = \gamma$$

where

$$\gamma \text{ is the image of } \mathcal{N}(0, I_{\mathbb{R}^3}) \text{ by } (x, y, z) \mapsto (\chi^f(p))^{1/2} x + \sigma_p y z,$$

with

$$\sigma_p^2 = \sum_{k \in \mathbb{Z}^d} (P(0 \in I \text{ and } k \in I) - \theta(p)^2),$$

where I is the infinite component of G .

In this case γ is a Gaussian measure when $p < p_c$ whereas γ is a Gaussian chaos of order 2 for $p > p_c$.

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